From Screw Theory to Compliant Mechanisms A Unified Framework for Mobility Analysis and Synthesis

Celebrate Ball's Screw Theory 150-Year Anniversary Special Day Event

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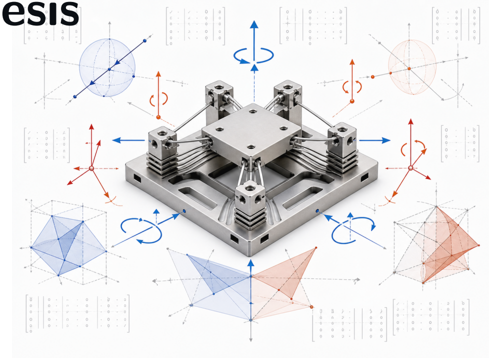
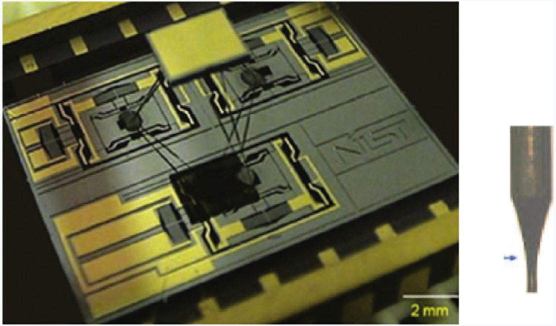


IFToMM Summer Screws 2026 (hybrid)
University College Cork, Cork, Ireland | July 22, 2026

Invited keynote (online) | 11:55–12:25 | 25-minute talk + 5-minute Q/A

Compliant mechanisms: motion without joints



NIST flexure-based hexapod nanopositioner (physical prototype).

0
backlash

0
frictional joints

1
monolithic body

The design question

Make the desired motion **easy**; make parasitic motion **expensive**.

- Precision through elastic deformation
- Function encoded by stiffness
- Mobility without discrete joints

Image: Fig. 1(a), Shi, Su and Dagalakis, *Mechanism and Machine Theory*, 80:246–264, 2014; NIST-hosted publication.

Mobility is a subspace before it is a number

Rigid-link baseline

$$M = \lambda(n - 1 - j) + \sum_i f_i$$

A compliant mechanism asks more

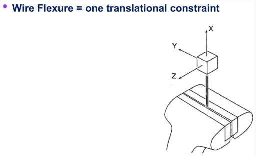
- Which translation or rotation?
- Where is the motion axis?
- Is the screw pitch finite?
- How large is the stiffness gap?

dimension gives the count

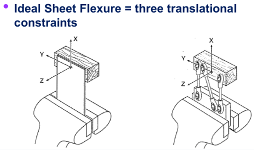
basis gives the design meaning

Rigid mobility count: Waldron, Kinzel and Agrawal, 2016. Compliant-space interpretation: Su, *J. Mechanisms and Robotics*, 3:041010, 2011; Zhang, Su and Liao, *Mechanism and Machine Theory*, 79:80–93, 2014.

Constraint-based design: place stiffness deliberately



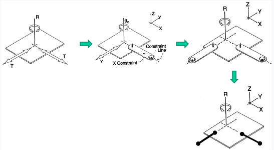
Wire: one dominant constraint



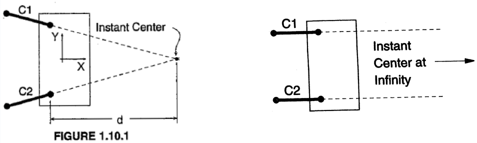
Sheet: three dominant constraints

Geometry creates separation

$$\frac{k_x}{k_z} = 4\left(\frac{l}{t}\right)^2, \qquad \frac{k_y}{k_z} = \left(\frac{w}{t}\right)^2$$



Intersecting constraints leave rotation.



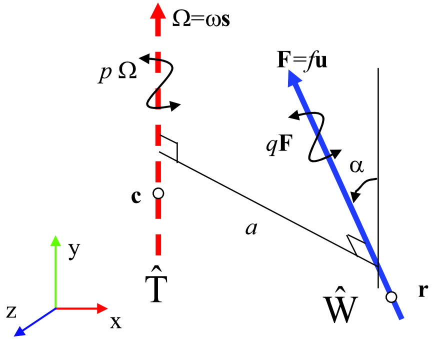
Parallel constraints leave translation.

Constraint-based design: Blanding, *Exact Constraint*, 1999; flexure interpretation from the lecture notes.

Ball's 1876 idea still organizes modern design

1876 — Ball's treatise
2009 — Type synthesis
2011 — Mobility algebra
2014 — Finite stiffness
2026 — Unified loop

A screw unifies **rotation**, **translation**, axis location, and pitch in one geometric object.



Historical anchor: Ball, *A Treatise on the Theory of Screws*, 1876/1900. Modern compliant-mechanism bridge: Su, Dorozhkin and Vance, ASME DETC2009-86684, 2009.

Twists describe motion; wrenches describe action

Motion screw: twist

$$\widehat{T} = \begin{bmatrix} \cdot \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{r} \times \mathbf{s} + h\mathbf{s} \end{bmatrix}$$

- axis direction $\mathbf{s}$
- axis point $\mathbf{r}$
- pitch h

Action screw: wrench

$$\widehat{W} = \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{u} \\ \mathbf{r} \times \mathbf{u} + h_w \mathbf{u} \end{bmatrix}$$

- force direction $\mathbf{u}$
- line of action $\mathbf{r}$
- wrench pitch h_w

freedom space = span of twists

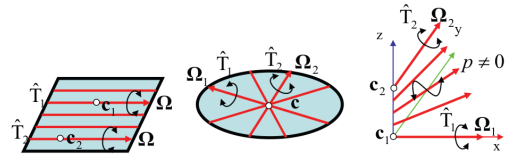
constraint space = span of wrenches

Notation follows the lecture notes and Ball, *A Treatise on the Theory of Screws*, 1876/1900.

Reciprocity converts desired freedom into required constraint

Zero virtual work

$$\widehat{T}\circ\widehat{W} = \mathbf{V}^{\mathsf{T}}\mathbf{F} + {\cdot}^{\mathsf{T}}\mathbf{M} = \widehat{T}^{\mathsf{T}}\boldsymbol{\Delta}\widehat{W} = 0$$



(a) two parallel lines (b) two intersecting lines (c) two skew lines

The same DOF count can represent different screw systems.

$$\mathcal{C} = \mathrm{null}(\mathbf{T}^{\mathsf{T}}\boldsymbol{\Delta}), \qquad \mathcal{F} = \mathrm{null}(\mathbf{W}^{\mathsf{T}}\boldsymbol{\Delta})$$

$$\dim\mathcal{F} + \dim\mathcal{C} = 6$$

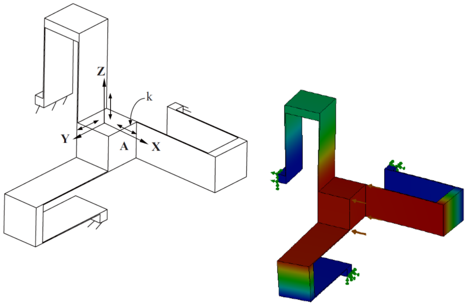
Reciprocal-space design and geometry: Su, Dorozhkin and Vance, ASME DETC2009-86684, 2009.

01 | ANALYSIS

Mobility analysis

For a given mechanism,
what is its freedom space?

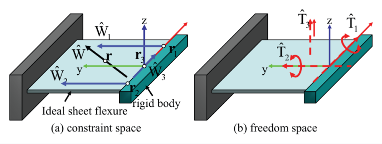
How many DOF does this CM have?



Mobility analysis identifies the dimension, directions, and locations of the allowable motion screws.

Constraint-pattern analysis: encode the sheet as wrenches

ANALYSIS



Coordinate choice

Sheet plane: xy, normal: z. The two parallel y-constraints pass through $x = \pm 1$.

Each ideal constraint is a line-force wrench

$$\widehat{W}_i = \begin{bmatrix} \mathbf{F}_i \\ \mathbf{r}_i \times \mathbf{F}_i \end{bmatrix}$$

$$(\mathbf{r}_1, \mathbf{F}_1) = (\mathbf{e}_x, \mathbf{e}_y), \quad (\mathbf{r}_2, \mathbf{F}_2) = (-\mathbf{e}_x, \mathbf{e}_y),$$
$$(\mathbf{r}_3, \mathbf{F}_3) = (\mathbf{0}, \mathbf{e}_x).$$

$$\mathbf{W} = [\, \widehat{W}_1 \ \widehat{W}_2 \ \widehat{W}_3 \,] = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right], \qquad \operatorname{rank} \mathbf{W} = 3.$$

Ideal sheet-flexure constraint space: Su, Dorozhkin and Vance, ASME DETC2009-86684, 2009.

Reciprocity reveals the complementary freedom space

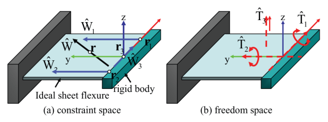
Solve the reciprocal null space

$$\widehat{T} = \begin{bmatrix}\theta_x & \theta_y & \theta_z & \delta_x & \delta_y & \delta_z\end{bmatrix}^{\mathsf{T}}, \qquad \mathbf{W}^{\mathsf{T}}\boldsymbol{\Delta}\widehat{T} = \mathbf{0}.$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix}\theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$\delta_y + \theta_z = 0, \quad \delta_y - \theta_z = 0, \quad \delta_x = 0$$

$$\Longrightarrow \quad \delta_x = \delta_y = \theta_z = 0.$$



$$\mathbf{T} = [\,\hat{R}_x \;\; \hat{R}_y \;\; \hat{P}_z\,] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

$$\operatorname{rank}\mathbf{T} = 3$$

two rotations **normal translation**

Constraint-pattern analysis and reciprocal freedom basis: Su, Dorozhkin and Vance, ASME DETC2009-86684, 2009.

Every flexure enters as a screw-system primitive

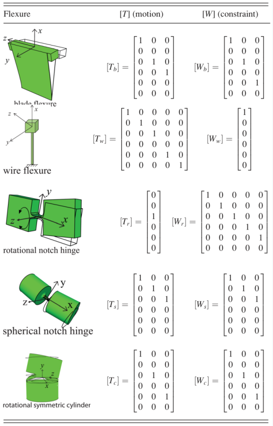
ANALYSIS

Element interface

$$(\mathbf{T}_e, \mathbf{W}_e), \qquad \mathbf{T}_e^{\mathsf{T}} \Delta \mathbf{W}_e = \mathbf{0}$$

- geometry stays local
- motion and constraint are dual
- rank records independence

blade wire notch



Flexure	$[T]$ (motion)	$[W]$ (constraint)
blade flexure	$[T_b] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$[W_b] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
wire flexure	$[T_w] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	$[W_w] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
rotational notch hinge	$[T_r] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$[W_r] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
spherical notch hinge	$[T_s] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$[W_s] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
rotational symmetric cylinder	$[T_c] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$[W_c] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Flexure screw-system library: Su, *J. Mechanisms and Robotics*, 3:041010, 2011.

Analyze top-down; assemble bottom-up

Decompose mechanism into flexure modules → **Model** assign local T_e, W_e → **Transform** express all screws in one frame → **Assemble** concatenate by topology

Geometry enters here

$$\mathrm{Ad}(\mathbf{R},\mathbf{d}) = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ [\mathbf{d}]_\times \mathbf{R} & \mathbf{R} \end{bmatrix}$$

One common frame

$$\widehat{T}' = \mathrm{Ad}\;\widehat{T}, \qquad \widehat{W}' = \mathrm{Ad}^{-\mathsf{T}}\;\widehat{W}$$

Without this step, rank tests mix incompatible coordinates.

Mobility workflow: Su, *J. Mechanisms and Robotics*, 3:041010, 2011. Coordinate transforms are the same adjoint operators used in compliance assembly: Su, Shi and Yu, *J. Mechanical Design*, 134, 2012.

Serial chains add motion spaces

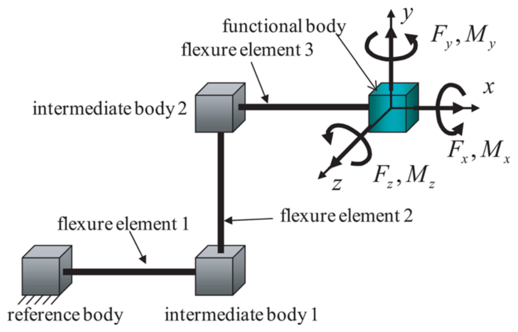
ANALYSIS

Serial assembly

$$\mathbf{T}_{\mathrm{ser}} = \begin{bmatrix} \mathrm{Ad}_1\,\mathbf{T}_1 & \mathrm{Ad}_2\,\mathbf{T}_2 & \cdots & \mathrm{Ad}_m\,\mathbf{T}_m \end{bmatrix}$$

$$f_{\mathrm{ser}} = \mathrm{rank}(\mathbf{T}_{\mathrm{ser}})$$

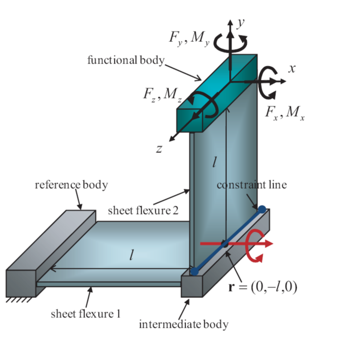
- concatenate candidate freedoms
- remove dependent columns
- retain the full twist basis



A serial chain contributes the union of its transformed motion spaces.

Serial mobility assembly: Su, *J. Mechanisms and Robotics*, 3:041010, 2011.

Serial example: two blades give five independent freedoms



Each ideal blade

$$\mathrm{rank}(\mathbf{T}_1) = \mathrm{rank}(\mathbf{T}_2) = 3$$

After transformation and reduction

$$\mathrm{rank}([\mathrm{Ad}_1\,\mathbf{T}_1 \quad \mathrm{Ad}_2\,\mathbf{T}_2]) = 5$$

$$3 + 3 \neq 6$$

One candidate motion is already in the span of the others.

Two-blade worked example and rank method: Su, *J. Mechanisms and Robotics*, 3:041010, 2011.

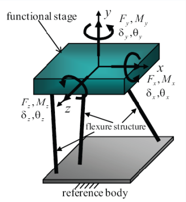
Parallel chains add constraint spaces

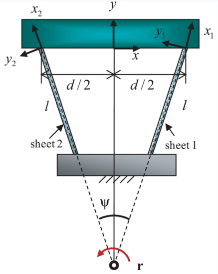
ANALYSIS

Parallel assembly

$$\mathbf{W}_{\mathrm{par}} = \left[\mathrm{Ad}_1^{-\mathsf{T}} \mathbf{W}_1 \quad \cdots \quad \mathrm{Ad}_m^{-\mathsf{T}} \mathbf{W}_m \right]$$

$$c_{\mathrm{par}} = \mathrm{rank}(\mathbf{W}_{\mathrm{par}}), \qquad \mathcal{F}_{\mathrm{par}} = \mathrm{null}(\mathbf{W}_{\mathrm{par}}^{\mathsf{T}} \boldsymbol{\Delta})$$



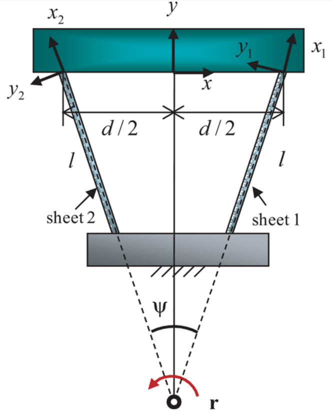


constraints combine

freedom survives

Parallel mobility assembly: Su, *J. Mechanisms and Robotics*, 3:041010, 2011.

Parallel example: one DOF has a location



Five independent constraints

$$\mathrm{rank}(\mathbf{W}_{\mathrm{par}}) = 5 \quad \Longrightarrow \quad \dim \mathcal{F} = 1$$

The surviving twist

$$\widehat{T}_{\angle} = \begin{bmatrix} 0 & 0 & 1 & -\frac{d}{2}\cot\left(\frac{\psi}{2}\right) & 0 & 0 \end{bmatrix}^{\mathsf{T}}$$

Rank gives the count.

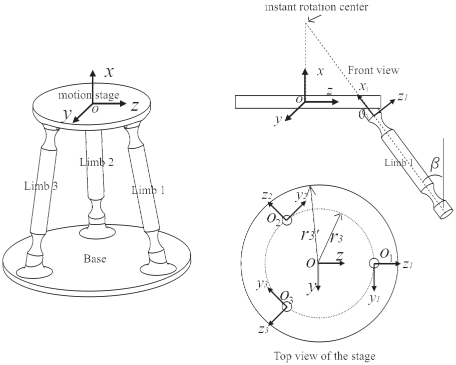
The twist gives the remote center.

Two-blade pivot: Su, *J. Mechanisms and Robotics*, 3:041010, 2011.

02 | QUANTITATIVE

Mobility from stiffness

Ideal constraints become finite stiffness ratios



Generalized eigentwist/eigenwrench decomposition of $\mathbf{C} = \mathbf{K}^{-1}$. Source: Zhang, Su and Liao, *Mechanism and Machine Theory*, 79:80–93, 2014.

From stiffness to six compliant screw axes

QUANTITATIVE

Engineering SVD view

$$\widetilde{\mathbf{K}} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^{\mathsf{T}}, \qquad \mathbf{C} = \mathbf{K}^{-1}$$

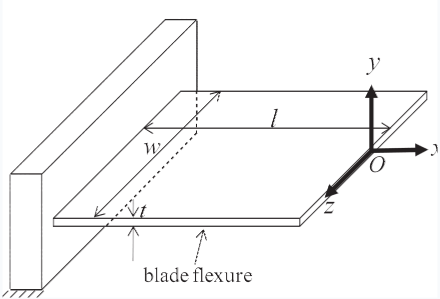
small $\sigma_i \Longrightarrow$ compliant screw direction

Generalized compliance decomposition

$$\mathbf{C}\widehat{W}_{\delta i} = c_{\delta i}\mathbf{T}_1\widehat{W}_{\delta i}, \qquad \widehat{T}_{\delta i} = \mathbf{T}_1\widehat{W}_{\delta i}$$
$$\mathbf{C}\mathbf{T}_1\widehat{T}_{\theta i} = c_{\theta i}\widehat{T}_{\theta i}$$

Normalized symmetric stiffness: singular/eigen directions identify soft versus stiff axes.

Generalized eigentwist/eigenwrench decomposition: Zhang, Su and Liao, *Mechanism and Machine Theory*, 79:80–93, 2014.



3 translational axes

3 rotational axes

A spectral gap turns six finite compliances into N freedoms

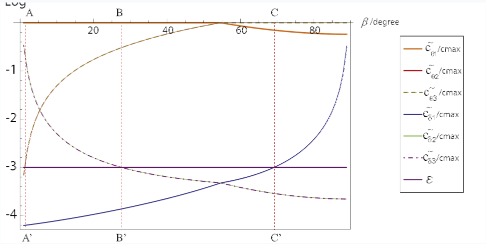
Make translation and rotation comparable

$$\widetilde{c}_{\delta i} = c_{\delta i}, \qquad \widetilde{c}_{\theta i} = l_c^2 c_{\theta i}$$

Order, normalize, classify

$$\widetilde{c}_1 \geq \cdots \geq \widetilde{c}_6, \qquad CR_i = \frac{\widetilde{c}_i}{\widetilde{c}_1}$$

$$m = \#\{CR_i \leq \epsilon\}, \qquad \boxed{N = 6 - m}$$



Geometry changes the spectral gap.

$$N = 3 \rightarrow 5 \rightarrow 3 \rightarrow 4$$

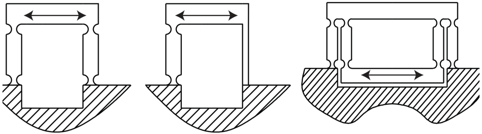
Typical first pass: $\epsilon = 10^{-3}$ to 10^{-2}.

Scaled-compliance mobility criteria: Zhang, Su and Liao, *Mechanism and Machine Theory*, 79:80–93, 2014.

03 | SYNTHESIS

Mobility synthesis

Desired freedom space ⟶ realizable flexure architecture



(a) (b) (c)

Reciprocity selects constraints; element equivalence expands the design catalog. Sources: Su, Dorozhkin and Vance, ASME DETC2009-86684, 2009; Su and Yue, *Mechanical Sciences*, 4:263–277, 2013.

Synthesis is mobility analysis run backward

SYNTHESIS

Specify the target

$$\mathcal{F}_d = \mathrm{range}(\mathbf{T}_d)$$
$$\mathcal{C}_d = \mathrm{null}(\mathbf{T}_d^{\mathsf{T}}\boldsymbol{\Delta})$$

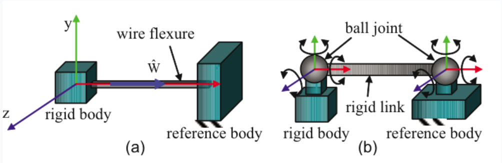
Verify the realization

$$\mathrm{range}(\mathbf{W}) = \mathcal{C}_d,$$
$$\mathrm{rank}(\mathbf{W}) = 6 - n$$

1. **Specify $\mathbf{T}_d$**
2. **Compute** reciprocal constraints
3. **Restrict** to realizable primitives
4. **Locate** independent screws
5. **Verify** rank and reciprocal null space
6. **Size and validate**

Reciprocal-space synthesis: Su, Dorozhkin and Vance, ASME DETC2009-86684, 2009.

A wire is a line screw



pure-force wrench
along the wire axis

Wire constraint wrench

$$\widehat{W}_i = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{r}_i \times \mathbf{u}_i \end{bmatrix}$$

Line-screw condition

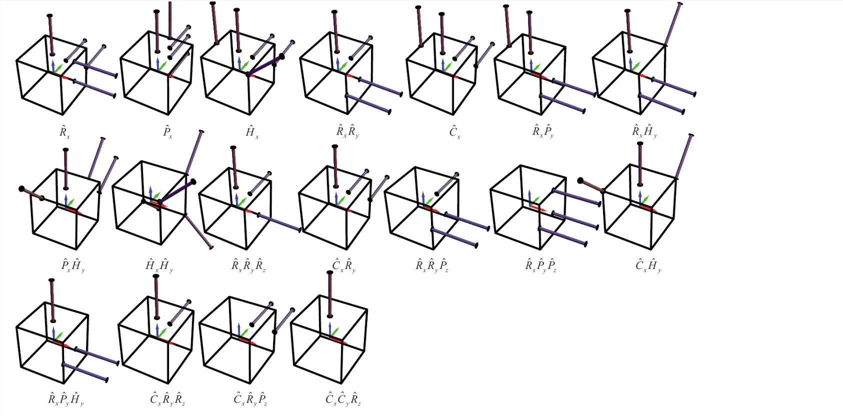
$$\mathbf{F}^{\mathsf{T}}\mathbf{M} = 0, \qquad \mathbf{F} \neq \mathbf{0}$$

Necessary design test

The reciprocal space must contain $6 - n$ independent line wrenches.

Wire line-screw realizability: Su and Tari, *J. Mechanical Design*, 132:121002, 2010.

Typical wire-flexure designs span 1–5 DOF



18 realizable spaces **1–5 DOF** **worked next:** $H_x H_y$

Fig. 3: 18 realizable orthogonal motion spaces, 1–5 DOF: Su and Tari, *J. Mechanical Design*, 132:121002, 2010.

Worked synthesis: prescribe two helical freedoms

1. Desired motion space

$$\mathcal{F}_d = \mathrm{span}\{\widehat{H}_x, \widehat{H}_y\}, \qquad \mathbf{T}_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ p_x & 0 \\ 0 & p_y \\ 0 & 0 \end{bmatrix}.$$

2. Enforce reciprocity

For $\widehat{W} = [\mathbf{F}; \mathbf{M}]$,

$$\mathbf{T}_d^{\mathsf{T}} \boldsymbol{\Delta} \widehat{W} = \begin{bmatrix} M_x + p_x F_x \\ M_y + p_y F_y \end{bmatrix} = \mathbf{0},$$

$$M_x = -p_x F_x, \qquad M_y = -p_y F_y.$$

3. Restrict each constraint to a wire

$$\mathbf{F}^{\mathsf{T}}\mathbf{M} = 0 \implies -p_x F_x^2 - p_y F_y^2 + F_z M_z = 0,$$
$$F_x^2 + F_y^2 + F_z^2 \neq 0.$$

4. Choose four solutions

i	F_x	F_y	F_z	M_x	M_y	M_z
1	1	0	1	$-p_x$	0	p_x
2	1	0	-1	$-p_x$	0	$-p_x$
3	0	1	1	0	$-p_y$	p_y
4	0	1	-1	0	$-p_y$	$-p_y$

Four reciprocal line wrenches are obtained.

Two-DOF helical-motion synthesis, Sec. 4.3: Su and Tari, *J. Mechanical Design*, 132:121002, 2010.

Assemble, verify, and place the four wires

Assembled constraint matrix

$$\mathbf{W}_{H_xH_y} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -p_x & -p_x & 0 & 0 \\ 0 & 0 & -p_y & -p_y \\ p_x & -p_x & p_y & -p_y \end{bmatrix}.$$

Synthesis proof ($p_xp_y \neq 0$)

$$\mathbf{T}_d^{\mathsf{T}} \boldsymbol{\Delta} \mathbf{W}_{H_xH_y} = \mathbf{0}, \quad \mathbf{F}_i^{\mathsf{T}} \mathbf{M}_i = 0,$$

$$\operatorname{rank}(\mathbf{W}_{H_xH_y}) = 4.$$

$$\operatorname{null}(\mathbf{W}_{H_xH_y}^{\mathsf{T}} \boldsymbol{\Delta}) = \operatorname{range}(\mathbf{T}_d) = \operatorname{span}\{\widehat{H}_x, \widehat{H}_y\}.$$

Recover each physical wire line

$$\mathbf{M}_i = \mathbf{r}_i \times \mathbf{F}_i, \qquad \mathbf{r}_i = \frac{\mathbf{F}_i \times \mathbf{M}_i}{\|\mathbf{F}_i\|^2}.$$

i	$\mathbf{F}_i$	$\mathbf{r}_i$
1	$(1, 0, 1)$	$(0, -p_x, 0)$
2	$(1, 0, -1)$	$(0, p_x, 0)$
3	$(0, 1, 1)$	$(p_y, 0, 0)$
4	$(0, 1, -1)$	$(-p_y, 0, 0)$

Physical result

Four parallel wires along these lines give $6 - \operatorname{rank}(\mathbf{W}_{H_xH_y}) = 2$ and leave exactly the H_xH_y motion space.

Line-wrench realization of H_xH_y: Su and Tari, *J. Mechanical Design*, 132:121002, 2010.

Type synthesis of a compliant P_x joint

SYNTHESIS

1. Prescribe the freedom

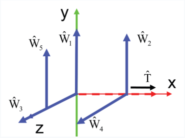
$$\widehat{T}_P = \begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}^{\mathsf{T}}$$

$$\widehat{T}_P^{\mathsf{T}} \boldsymbol{\Delta} \widehat{W} = 0 \implies F_x = 0$$

2. Restrict to line-force constraints

$$\widehat{W} = \begin{bmatrix}0 & f_y & f_z & m_x & m_y & m_z\end{bmatrix}^{\mathsf{T}}$$

$$\mathbf{F}^{\mathsf{T}}\mathbf{M} = f_y m_y + f_z m_z = 0, \qquad f_y^2 + f_z^2 \neq 0$$



$$\mathbf{W}_P = [\,\widehat{W}_1\ \widehat{W}_2\ \widehat{W}_3\ \widehat{W}_4\ \widehat{W}_5\,] = \begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0\end{bmatrix}.$$

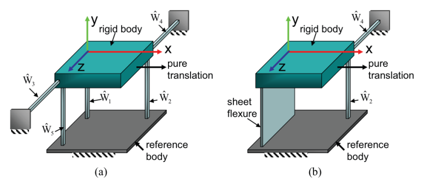
3. Verify the synthesized mobility

$$\operatorname{rank}(\mathbf{W}_P) = 5,$$

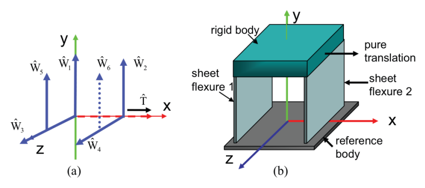
$$\operatorname{null}(\mathbf{W}_P^{\mathsf{T}}\boldsymbol{\Delta}) = \operatorname{span}\{\widehat{T}_P\}.$$

Compliant prismatic-joint type synthesis: Su, Dorozhkin and Vance, ASME DETC2009-86684, 2009.

One P_x screw space, three physical realizations



Five wires $\longrightarrow$ one sheet + two wires



Redundant $\widehat{W}_6 \longrightarrow$ two parallel sheets

Equivalent constraint-space substitutions

$$\mathbf{W}_P = [\,\widehat{W}_1 \cdots \widehat{W}_5\,],$$

$$\mathcal{C}_{S_1} = \text{span}\{\widehat{W}_1, \widehat{W}_3, \widehat{W}_5\}.$$

The remaining $\widehat{W}_2, \widehat{W}_4$ stay as wire flexures.

Benign redundancy preserves P_x

$$\widehat{W}_6 = \begin{bmatrix} 0 & 1 & 0 \mid -1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$
$$= \widehat{W}_2 + \widehat{W}_5 - \widehat{W}_1$$

$$\text{rank}[\,\mathbf{W}_P\ \widehat{W}_6\,] = 5,$$
$$\text{null}\Big([\,\mathbf{W}_P\ \widehat{W}_6\,]^{\mathrm{T}}\boldsymbol{\Delta}\Big) = \text{span}\{\widehat{T}_P\}.$$

Equivalent wire and sheet-flexure realizations: Su, Dorozhkin and Vance, ASME DETC2009-86684, 2009.

Flexure primitives: motion and constraint spaces

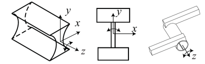
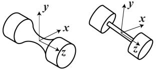
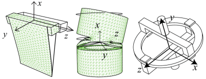
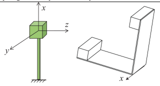
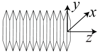
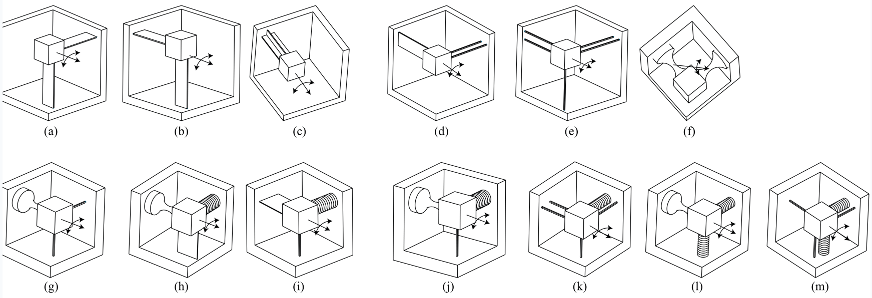
Flexure	Freedom Symbol	[T]	[W]
 Notch/Living Hinge, Short Beam, Split Tube	R	$[\hat{R}_z]$	$[\hat{F}_x \quad \hat{F}_y \quad \hat{F}_z \quad \hat{M}_x \quad \hat{M}_y]$
 Spherical Notch, Short Wire/Rod	S=3R	$[\hat{R}_x \quad \hat{R}_y \quad \hat{R}_z]$	$[\hat{F}_x \quad \hat{F}_y \quad \hat{F}_z]$
 Blade/Sheet/ Leaf Spring, Rotational Symmetric Cylinder, Disc Coupling	B=2R+P	$[\hat{R}_x \quad \hat{R}_z \quad \hat{P}_y]$	$[\hat{F}_x \quad \hat{F}_z \quad \hat{M}_y]$
 Long Wire/Rod, Corner Blade	W=3R-2P	$[\hat{R}_x \quad \hat{R}_y \quad \hat{R}_z \quad \hat{P}_y \quad \hat{P}_z]$	$[\hat{F}_x]$
 Bellow Spring	B_s=2R-3P	$[\hat{R}_x \quad \hat{R}_y \quad \hat{P}_x \quad \hat{P}_y \quad \hat{P}_z]$	$[\hat{M}_z]$

Table 1: primitive freedom and constraint spaces: Su and Yue, *Mechanical Sciences*, 4:263–277, 2013.

R-joint catalog: same motion, different embodiments



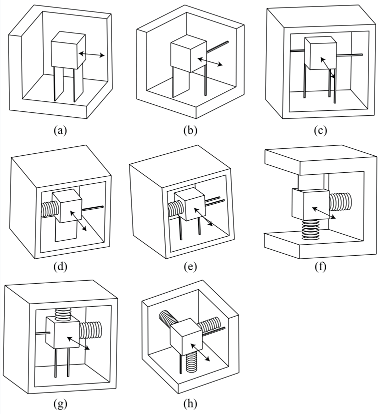
Equivalence principle

$$\mathrm{range}(\mathbf{T}_{\mathrm{alt}}) = \mathrm{range}(\mathbf{T}_d)$$

- blades, wires, notches, bellows
- exact or deliberately redundant constraints
- choose embodiment after the subspace

Element catalogs: Su and Yue, *Mechanical Sciences*, 4:263–277, 2013.

P-joint catalog: eight parallel realizations



Without bellows springs

(a) $2B$ **(b)** B-$2W$ **(c)** $5W$

With bellows springs B_s

(d) B_s-B-W **(e)** B_s-$4W$
(f) $2B_s$-B **(g)** $2B_s$-$3W$
(h) $3B_s$-$2W$

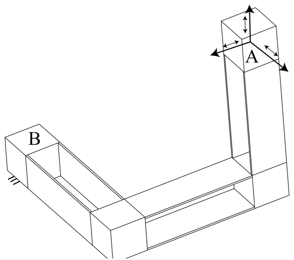
Equivalent freedom spaces

Every architecture removes five independent motions and preserves the translation indicated by the arrow:

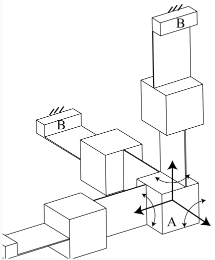
$$\mathrm{range}(\mathbf{T}_a) = \cdots = \mathrm{range}(\mathbf{T}_h) = \mathrm{span}\{\widehat{P}\}.$$

Fig. 5: eight parallel P-joint designs: Su and Yue, *Mechanical Sciences*, 4:263–277, 2013.

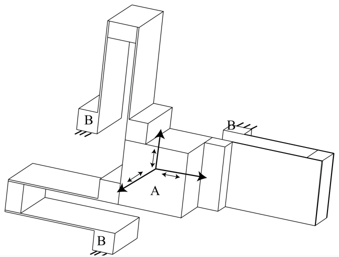
Complex flexures follow the same two topology rules



Serial ***PPP***
adds three translation spaces



Parallel ***P*****-constraints**
remove translations; leave rotations



Parallel ***R*****-constraints**
remove rotations; leave translations

Serial adds freedom **Parallel adds constraint**

Element composition examples: Su and Yue, *Mechanical Sciences*, 4:263–277, 2013.

Conclusions

1 Constraint-based design

Place high-stiffness directions deliberately; the complementary subspace is the intended motion.

2 Screw theory is the algebraic bridge

Twists encode freedoms, wrenches encode constraints, and reciprocity formalizes complementary patterns.

3 Mobility analysis

Known elements and topology $\rightarrow$ transformed screw bases $\rightarrow$ rank and reciprocal null space.

4 Mobility synthesis

Desired twist space $\rightarrow$ reciprocal constraints $\rightarrow$ realizable flexure architecture.

desired motion freedom space → **screw reciprocity** constraint space → **topology** physical flexures → **stiffness check** finite geometry

Questions? su.298@osu.edu

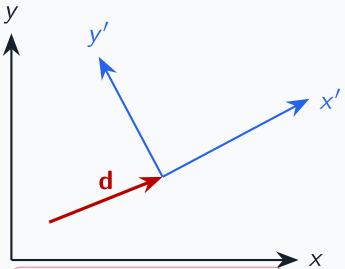
Backup: adjoint transformation in one line

Rigid transform $(\mathbf{R}, \mathbf{d})$

$$\mathrm{Ad}(\mathbf{R}, \mathbf{d}) = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ [\mathbf{d}]_\times \mathbf{R} & \mathbf{R} \end{bmatrix}$$

Dual transformation

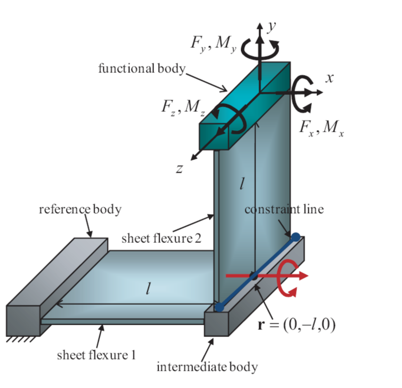
$$\widehat{T}' = \mathrm{Ad}\, \widehat{T}, \qquad \widehat{W}' = \mathrm{Ad}^{-\mathsf{T}}\, \widehat{W}$$



All columns must describe the same stage frame before rank or null-space tests.

Coordinate convention used throughout Su, *J. Mechanisms and Robotics*, 3:041010, 2011; Su, Shi and Yu, *J. Mechanical Design*, 134, 2012.

Backup: serial two-blade rank reduction



Assembled candidate matrix

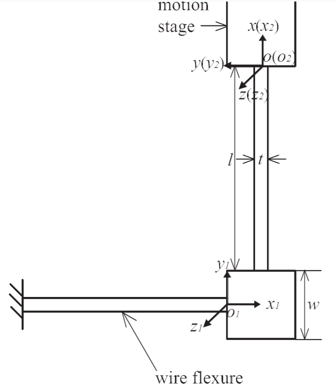
$$\mathbf{T}_{\text{ser}} = [\mathrm{Ad}_1\,\mathbf{T}_b \quad \mathrm{Ad}_2\,\mathbf{T}_b] \in \mathbb{R}^{6\times 6}$$

Row/column reduction

$$\mathbf{T}_{\text{ser}} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank} = 5$$

Two-blade rank reduction: Su, *J. Mechanisms and Robotics*, 3:041010, 2011.

Backup: Criterion 2 needs an absolute compliance scale



When no relative spectral gap exists

$$m = 0$$

$$N = \begin{cases} 0, & \widetilde{c}_1 \leq c_\epsilon \\ 6, & \widetilde{c}_1 > c_\epsilon \end{cases}$$

- relative ratios alone are ambiguous
- absolute compliance is application-dependent
- load scale belongs in the mobility decision

Criterion 2: Zhang, Su and Liao, *Mechanism and Machine Theory*, 79:80–93, 2014.

Backup: extension to 3-DOF translational compliant platforms

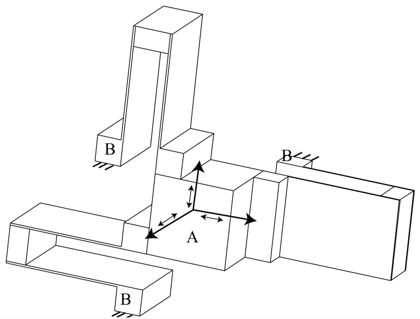
Desired platform freedom

$$\mathbf{T}_d = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_3 \end{bmatrix}$$

Reciprocal platform constraints

$$\mathcal{C}_d = \text{span}\{\text{three pure couples}\}$$

1. enumerate admissible limbs
2. satisfy assembly geometry
3. substitute equivalent flexures
4. verify rank and spectral gap



Three-translation type synthesis: Yue, Zhang, Su and Kong, *J. Mechanisms and Robotics*, 7:031012, 2015.

Backup: selected publications behind the framework

Mobility analysis and synthesis

- H.-J. Su, D. Dorozhkin and J. Vance, screw-theory type synthesis, ASME DETC2009-86684.
- H.-J. Su and H. Tari, orthogonal motions with parallel wire flexures, *JMD*, 2010.
- H.-J. Su, mobility analysis via screw algebra, *JMR*, 2011.
- H.-J. Su and C. Yue, freedom/constraint element catalogs, *Mechanical Sciences*, 2013.

Quantitative compliance and extensions

- H.-J. Su, H. Shi and J. Yu, symbolic compliance analysis and synthesis, *JMD*, 2012.
- Y. Zhang, H.-J. Su and Q. Liao, mobility criteria from compliance decomposition, *MMT*, 2014.
- C. Yue, Y. Zhang, H.-J. Su and X. Kong, 3-DOF translational compliant parallel mechanisms, *JMR*, 2015.